A coupled Landau model describing the Strouhal–Reynolds number profile of a three-dimensional circular cylinder wake

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(Received 17 March 2003; accepted 5 May 2003; published 23 July 2003)

A known bifurcation scenario describing the development and interaction of Mode A and Mode B vortex shedding modes of a circular cylinder wake is extended to predict the Strouhal–Reynolds number profile over the three-dimensional transitions. The mode amplitudes are described by coupled Landau equations and, with frequency information being included by the addition of complex coefficients, the model predicts the discontinuous nature of the Strouhal–Reynolds number shedding profile of the circular cylinder wake throughout the laminar three-dimensional transition regime. The model coefficients are determined from computations of the three-dimensional modes of a circular cylinder wake. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1597471]

The transition to three-dimensionality in the wake of bluff bodies is of great importance to myriad physical engineering and scientific problems, and as such, the phenomenon has been afforded much attention through a great body of research over many decades.

The canonical problem for research into bluff body wake flows has been the flow around a straight circular cylinder. Williamson,[1] in 1988, identified two stages in the transition to three-dimensionality of the vortex-shedding wake behind the circular cylinder. The stages are characterized by discontinuous transitions in the Strouhal–Reynolds number profile, coupled with the evolution of spanwise periodic deformations of the two-dimensional vortex-shedding street. The first transition, resulting in a discontinuous and hysteretic reduction in the Strouhal frequency of shedding, has become known as Mode A. The Mode A wake sees the inception of streamwise vortex loops in the braid region between successive rollers in the vortex shedding street, and the spanwise wavelength of the repeating three-dimensional structures is approximately four cylinder diameters $(4d)$. The transition to Mode A occurs over a hysteretic Reynolds number range of approximately $180<\mathrm{Re}<190$, and the subsequent transition to the Mode B wake occurs gradually over an approximate Reynolds number range $230<\mathrm{Re}<265$.[2] The Mode B wake pattern has a much shorter spanwise wavelength (approximately $1d$), and the transition to Mode B sees a gradual transfer of energy from Mode A to Mode B wake structures, coupled with an increase in the Strouhal frequency of shedding near to the continuation of the laminar two-dimensional Strouhal profile.

Three-dimensional numerical computations by Thompson, Hourigan, and Sheridan,[3] in 1996, provided striking visualizations of the three-dimensional structure and spanwise wavelength of the Mode A and Mode B wake structures, as well as divulging the spatiotemporal symmetry of the wakes. Visualizations from similar computations have been presented by Zhang *et al.*,[4] in 1995, and Henderson,[2] in 1997.

A landmark application of a linear Floquet stability analysis to the two-dimensional vortex shedding wake of a circular cylinder by Barkley and Henderson,[5] in 1996, gave predictions of the critical Reynolds numbers for the three-dimensional transitions pertaining to both Mode A and Mode B, as well as correctly identifying the respective spanwise wavelength and spatiotemporal symmetry characteristics of the transition modes. They predicted Mode A to first become unstable to perturbations with a spanwise wavelength of $3.96d$ at $\mathrm{Re}\approxeq188.5$, and the onset of Mode B to occur for perturbations with a spanwise wavelength of $0.822d$ at $\mathrm{Re}\approxeq259$.

The nonlinear behavior of the three-dimensional transition modes of the wake of the circular cylinder was first investigated by Henderson and Barkley,[6] in 1996. They determined the complex Landau coefficients and ascertained whether the modes occurred through supercritical or subcritical bifurcations. The linear coefficients of the Landau equation were in agreement with the growth rates determined from stability analysis, and an analysis of the cubic coefficients provided the criticality of the transitions. Consistent with previous observations, the Mode A transition was found to occur through a subcritical bifurcation, indicating a hysteretic transition. The Mode B transition was found to occur through a supercritical bifurcation, consistent with a nonhysteretic transition.

The Landau model has been applied successfully to various two- and three-dimensional transition modes in fluid mechanics applications. Provansal, Mathis, and Boyer,[7] in 1987, used the Landau equation to model the Hopf transition of a steady circular cylinder wake to an unsteady wake at around $\mathrm{Re}=48.6$. The criticality of the asymmetric regular and Hopf transitions of the wake of a sphere were also accurately determined, in 2001, by Thompson, Leweke, and Provansal,[8] and Ghidersa and Dušek.[9]

In 2000, Barkley, Tuckerman, and Golubitsky[10] suggested a bifurcation scenario consisting of coupled evolution equations for the amplitudes of the Mode A and Mode B instabilities in the wake of a circular cylinder. The coupled evolution equations

$$
\begin{aligned}
A_{n+1} &= \mu^A(\mathrm{Re})A_n + \alpha_1^A|A_n|^2A_n + \gamma_1^A|B_n|^2A_n \\
&\quad + \alpha_2^A|A_n|^4A_n, \\
B_{n+1} &= \mu^B(\mathrm{Re})B_n + \alpha_1^B|B_n|^2B_n + \gamma_1^B|A_n|^2B_n,
\end{aligned} \tag{1}
$$

are, essentially, truncated discrete Landau equations, incorporating additional coupling terms for A_n and B_n. A fifth order truncation is sufficient to model the subcritical onset of Mode A, and a third order truncation is sufficient to model the supercritical onset of Mode B. These Landau equations, incorporating third order coupling, are a normal form for the simultaneous bifurcation of Modes A and B. In order to model Strouhal frequency variation of the transition modes, the coefficients in Eq. (1) are expanded into the complex plane, and are evaluated for computed Strouhal frequencies of the saturated three-dimensional modes.

The values of the coefficients will be discussed later, however it is pertinent to note that A_n and B_n represent the complex amplitudes of Mode A and Mode B, respectively, for the nth oscillation period. The μ^A and μ^B coefficients are the real Floquet multipliers of the linear instabilities of the cylinder wake, and the α_1^A and α_1^B coefficients are the cubic coefficients of the Landau model from the Henderson[2] study. The α_2^A term is the additional quintic coefficient required to describe the saturation and hysteresis of Mode A, and finally the γ_1^A and γ_1^B coefficients determine the mode coupling of the system, and have been estimated from experimental observations[1] of the transition from Mode A to Mode B in the circular cylinder wake.

In order to incorporate temporal information into the coupled amplitude equations, we replace the evolution amplitudes A_n and B_n with complex amplitudes A and B, and in addition, the evolution equations are recast in the familiar differential equation form of the Landau equations. From the Floquet multiplier definition, $\mu \equiv \exp(\sigma T)$, we insert the linear growth rate coefficient (σ) into the Landau equations, where T is the period of oscillation of the two-dimensional shedding mode, giving

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= [\sigma^A(\mathrm{Re}) + \mathrm{i}\omega^A]A + \alpha_1^A(1+\mathrm{i}c_1^A)|A|^2A \\
&\quad + \gamma_1^A(1+\mathrm{i}d_1^A)|B|^2A + \alpha_2^A(1+\mathrm{i}c_2^A)|A|^4A, \\
\frac{\mathrm{d}B}{\mathrm{d}t} &= [\sigma^B(\mathrm{Re}) + \mathrm{i}\omega^B]B + \alpha_1^B(1+\mathrm{i}c_1^B)|B|^2B \\
&\quad + \gamma_1^B(1+\mathrm{i}d_1^B)|A|^2B.
\end{aligned} \tag{2}
$$

In Eq. (2), the angular frequency of the modes for infinitesimal amplitudes is given by $\omega^A = \omega^B = \omega \equiv 2\pi/T$, where T is the period of oscillation.

The linear complex coefficients, ω^A and ω^B, are functions of Reynolds number, and provide the angular oscillation frequency in the linear regime of the transition modes, corresponding to the laminar Strouhal number profile of the two-dimensional vortex shedding street. The complex coefficients, c_1^A, c_2^A, and c_1^B, determine the frequency behavior of the modes through saturation. The calculation of these coefficients is relatively simple, as they may be calculated independently for Mode A and Mode B. The c_1^B term is simply the Landau constant of the Mode B transition, and as we model the Mode B transition with a cubic truncation of the Landau model, the value of the coefficient may be determined by following the analysis of Dušek *et al.*,[11] in 1994, and Le Gal, Nadim, and Thompson,[12] in 2001. By assuming at saturation $B = \rho_B \exp(\mathrm{i}\Phi^B)$ in a cubic truncation of the Landau equation for Mode B alone, where $\rho_B = |B|$ and $\mathrm{d}\Phi^B/\mathrm{d}t = \omega^B$, the Landau constant can be expressed as a function of the global system parameters $c_1^B = (\omega^B - \omega_{\mathrm{sat}}^B)/\sigma^B$, where ω_{sat}^B is the saturated oscillation frequency. These global parameters are determined from a three-dimensional computation including only Mode B.

For calculation of the α_1^A, α_2^A, c_1^A, and c_2^A terms of the quintic Mode A transition model, two pairs of equations are formed for separate computations of the Mode A wake. By substituting $A = \rho_A \exp(\mathrm{i}\Phi^A)$ in the quintic Mode A Landau equation, neglecting the coupling term and grouping real and imaginary parts, the relationships

$$
\begin{aligned}
0 &= \sigma^A + \alpha_1^A\rho_A^2 + \alpha_2^A\rho_A^4, \\
0 &= \omega^A - \omega_{\mathrm{sat}}^A + \alpha_1^A c_1^A \rho_A^2 + \alpha_2^A c_2^A \rho_A^4,
\end{aligned} \tag{3}
$$

result for the sine and cosine coefficients.

The global parameters, ω^A, ω_{sat}^A, σ^A, and ρ_A, are determined for computations at two discrete Reynolds numbers in the Mode A transition regime, not far in excess of the critical Reynolds number, and the two pairs of equations are solved for the four unknown coefficients: α_1^A, α_2^A, c_1^A, and c_2^A. From the calculated values of α_1^A and α_2^A, the predicted Reynolds number range of hysteresis $(\Delta\,\mathrm{Re}^A)$ may be found from

$$
\Delta\,\mathrm{Re}^A = \frac{-(\alpha_1^A)^2}{4 m_A \alpha_2^A}, \tag{4}
$$

where m_A is the gradient of the growth rate, $\sigma^A(\mathrm{Re})$. The values determined for the present study predict $\Delta\,\mathrm{Re}^A \approxeq 16.2$, which is of the same order as the estimation of $\Delta\,\mathrm{Re}^A \approx 10$ from Henderson.[2] Note that this figure is solely determined from the pair of computations of Mode A above the critical Reynolds number.

The real coupling coefficients are determined by using the experimentally observed[1] Reynolds numbers for the first-occurring instance of Mode B, and the last-occurring instance of Mode A, in the cylinder wake. The coupled Mode A and Mode B equations are each evaluated at the Reynolds numbers at which their corresponding wake structures are last observed and first observed, respectively. For Mode A and Mode B, these critical Reynolds numbers are $\mathrm{Re}_{\mathrm{last}}^A \approx 260$ and $\mathrm{Re}_{\mathrm{first}}^B \approx 230$, respectively. Substituting $|A| = 0$ and $|B| = 0$ in the Mode A and Mode B equations, respectively, and solving for the coupled coefficients gives

TABLE I. Values of coefficients determined for the present model.

Coefficient	Value
σ^A	$1.699\times10^{-3}(\mathrm{Re}-187.41)$
σ^B	$4.868\times10^{-3}(\mathrm{Re}-258.24)$
ω^A and ω^B	$2\pi(8.539\times10^{-5}\,\mathrm{Re}+0.1999-4.009/\mathrm{Re})$
α_1^A	1.203×10^3
α_2^A	-1.313×10^7
α_1^B	-7.297×10^3
γ_1^A	-1.048×10^5
γ_1^B	1.033×10^3
c_1^A	-1.055
c_2^A	-0.3276
c_1^B	8.342×10^{-2}
d_1^A	-0.2
d_1^B	-0.25

$$\gamma_1^A=\frac{-\sigma^A_{\mathrm{Re}=\mathrm{Re}^A_{\mathrm{last}}}}{|B_{\mathrm{Re}=\mathrm{Re}^A_{\mathrm{last}}}|^2}=\frac{\alpha_1^B\sigma^A_{\mathrm{Re}=\mathrm{Re}^A_{\mathrm{last}}}}{\sigma^B_{\mathrm{Re}=\mathrm{Re}^A_{\mathrm{last}}}},$$
$$\gamma_1^B=\frac{-\sigma^B_{\mathrm{Re}=\mathrm{Re}^B_{\mathrm{first}}}}{|A_{\mathrm{Re}=\mathrm{Re}^B_{\mathrm{first}}}|^2}=\frac{2\alpha_2^A\sigma^B_{\mathrm{Re}=\mathrm{Re}^B_{\mathrm{first}}}}{\alpha_1^A+\sqrt{(\alpha_1^A)^2-4\alpha_2^A\sigma^A_{\mathrm{Re}=\mathrm{Re}^B_{\mathrm{first}}}}}. \tag{5}$$

The mode amplitude norms $|A|$ and $|B|$ in the present study are obtained from three-dimensional computations and are evaluated by the relationship

$$|A|\equiv\left[\Psi_{\mathrm{cylinder}}\int_\Omega|w^A|^2\mathrm{d}\Omega\right]^{1/2}, \tag{6}$$

where $|A|$ is the amplitude of the mode in question, Ψ_{cylinder} is a normalizing coefficient set to unity for simplicity, Ω is a cross section of the computational domain in the $x-y$ plane, and w^A is the spanwise Fourier coefficient of the w-velocity field corresponding to the wavelength of the mode in question. This amplitude quantity, while not strictly a global wake property (due to truncation of the mode at the computational domain outlet), has been employed successfully for circular cylinder wakes by Henderson,[2] in 1997, and Thompson, Leweke, and Provansal,[8] in 2001. Differences in both the integral chosen to evaluate the amplitude norm, and the choice of normalizing coefficient result in a quantitative difference between the values obtained for the present study, and the study of Barkley, Tuckerman, and Golubitsky,[10] as well as resulting in a large discrepancy between the present α and γ parameters, and previously obtained values.[10] The amplitude norm magnitudes are arbitrary, and the qualitative comparison between the present and previous work[10] is similar, suggesting that a high degree of computational accuracy exists for both studies.

Table I summarizes the values of the coefficients of the complex amplitude equations employed in this investigation. All real coefficients have been calculated from three-dimensional computations performed to complement the present study. The complex coefficients of first-order terms are derived from a parallel vortex shedding Strouhal–Reynolds number relationship determined from the present numerical calculations, employing the same form as the universal laminar Strouhal–Reynolds number profile proposed by Williamson,[13] in 1988.

The higher-order complex coefficients are determined from computations of the evolution and saturation of three-dimensional wakes corresponding to the Mode A and Mode B instabilities. Computations at Reynolds numbers Re=195 and Re=200 were employed to evaluate the coefficients c_1^A and c_2^A for the Mode A instability. A computation of the Mode B wake at Re=265 was used to evaluate the coefficient c_1^B. Values for the complex coupling coefficients, d_1^A and d_1^B, were chosen to equate the computed Strouhal frequencies at the last appearance of Mode A, and the onset of Mode B, respectively, with experimentally obtained frequencies.[1]

The coupled complex amplitude equations proposed here are solved simultaneously by employing a third order Adams–Bashforth scheme, giving

$$A_{i+1}=A_i+\frac{\Delta t}{12}[23f_i^A-16f_{i-1}^A+5f_{i-2}^A],$$
$$B_{i+1}=B_i+\frac{\Delta t}{12}[23f_i^B-16f_{i-1}^B+5f_{i-2}^B], \tag{7}$$

where f_i^A and f_i^B denote the right hand sides of the complex coupled Landau equations evaluated at the ith time step. The asymptotic frequency information may be evaluated from the saturated mode amplitudes directly, however, a future expansion of the present model intends to include a spanwise diffusion term to model long-wavelength three-dimensional wake patterns, in a fashion similar to the complex Ginzburg–Landau model applied to bluff ring wakes by Leweke and Provansal,[14] in 1995. The future expansion necessitates temporal integration of the model, and the Adams–Bashforth method is implemented here for the purpose of validation. The Adams–Bashforth method maintains an accuracy of order Δt^4, as verified by a brief temporal resolution study that determines the stability and accuracy of the present numerical formulation. Temporal stability was achieved for time steps $\Delta t\leqslant0.125$. The computed Strouhal frequency was used to monitor convergence of the model. At a time step of $\Delta t=0.125$, the computed Strouhal frequency was within 0.032% of the Strouhal frequency computed at $\Delta t=9.766\times10^{-4}$. The present study employs a time step of $\Delta t=0.1$, maintaining an accuracy of better than 0.025% and temporal convergence over the Reynolds number range being investigated.

The evolution equations of Barkley, Tuckerman, and Golubitsky[10] predict a bifurcation diagram showing three distinct three-dimensional mode branches for increasing Reynolds number. The present model predicts the same behavior, as displayed in Fig. 1. Notice the hysteresis at the onset of the Mode A branch for $180\lesssim\mathrm{Re}\lesssim188$, and the mixed A/B branch for $230\lesssim\mathrm{Re}\lesssim260$ as energy is transferred from Mode A to Mode B.

The present model evaluates the frequencies associated with the modes from the evolution equations in Eq. (1).[10] Strouhal frequencies are determined at increments of $\Delta\,\mathrm{Re}=1$ through the Reynolds number range $48<\mathrm{Re}<300$. Over

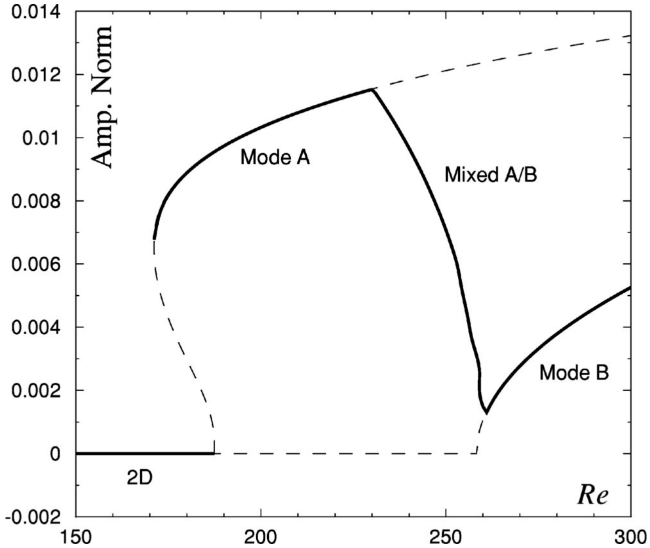

FIG. 1. Amplitude norm ($\sqrt{|A|^2+|B|^2}$) variation with Reynolds number, computed using the proposed coupled Landau model for Modes A and B. Dotted lines indicate the amplitudes of the uncoupled A and B branches. Time-averaged values are provided for 255<Re<260, due to fluctuation of the amplitude norm measurements in that range.

the mixed A/B regime ($230 \lesssim \mathrm{Re} \lesssim 260$), discrete oscillation frequencies were present for the Mode A and Mode B amplitudes. This corresponds to the discontinuous region of the Strouhal–Reynolds number profile of the circular cylinder wake where energy is transferred from Mode A wake structures to Mode B. The predicted Strouhal–Reynolds number profile from the present model is presented in Fig. 2. The experimental circular cylinder wake Strouhal–Reynolds

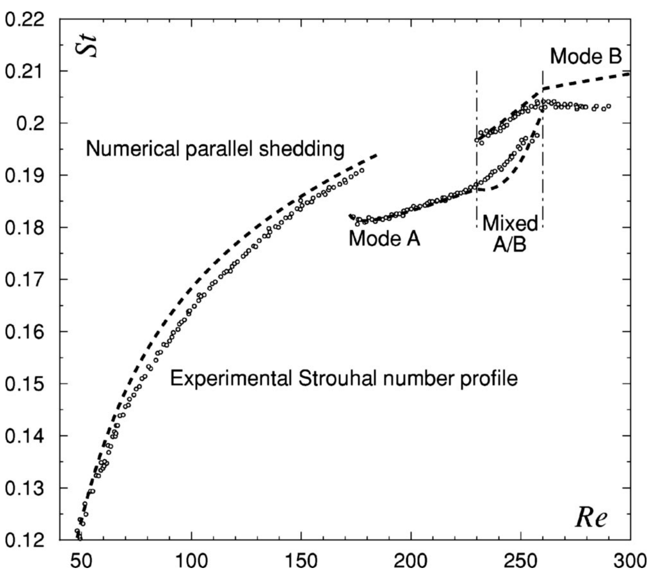

FIG. 2. A Strouhal–Reynolds number profile showing the computed Strouhal frequencies from the proposed coupled Landau model (indicated by dashed lines), and the experimental universal Strouhal curve determined by Williamson (Refs. 1 and 13) in 1988 (indicated by open circles).

number data, corrected for parallel shedding by Williamson,[1,13] in 1988, are included for comparison.

Despite only evaluating the complex model coefficients for Strouhal frequencies in the vicinity of the Mode A and Mode B transitions, the computations presented here show that a remarkable qualitative agreement is observed between the experimental Strouhal–Reynolds number profile of a circular cylinder wake through the three-dimensional transition regime, and the Strouhal frequencies determined using the present coupled Landau model. The shedding frequency of both Mode A and Mode B are very well predicted by the present model for $\mathrm{Re} \lesssim 260$. The constant Strouhal frequency of $\mathrm{St} \approx 0.203$, observed experimentally for $\mathrm{Re} \gtrsim 260$, differs from the increasing Strouhal frequency predicted by the model, probably due to longer-span instabilities that lower the shedding frequency from that of the pure Mode B wake computed for the coefficients of the present model. The two-dimensional shedding Strouhal profile also lies in good agreement with the corrected experimental data for parallel shedding.

ACKNOWLEDGMENTS

This work would not be possible without the computational resources provided by the Victorian Partnership for Advanced Computing (VPAC). This research has been supported by the Australian Postgraduate Award (APA) scholarship.

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